A nonlinear model of heat conduction

1992 J. Phys. A: Math. Gen. 25 939

(http://iopscience.iop.org/0305-4470/25/4/029)

J. Phys. A: Math. Gen. 25 (1992) 939-948. Printed in the UK

A nonlinear model of heat conduction

H Pascal

Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1

Abstract. A nonlinear model of heat propagation is presented, from which a new heat conduction equation is derived. An exact similarity solution in closed form of this equation is obtained, which reveals the travelling wave characteristics for the transient temperature distribution. It is shown that the temperature disturbances propagate with finite velocity, which is a monotonically decreasing function of time.

1. Introduction

It is well known that the standard heat conduction equation implies that a temperature disturbance will propagate with infinite velocity. According to the relativity theory this result is unacceptable because no disturbances can travel faster than light. From a continuum mechanics point of view no disturbances in a medium are likely to propagate faster than sound. Consequently, the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2}=a^2\frac{\partial T}{\partial t} \qquad a^2=\frac{\rho c}{\lambda} \tag{1}$$

fails to describe accurately the heat transfer mechanism over small times; the standard linear equation (1) belongs to a class known as parabolic equations.

In order to eliminate the paradox of infinite velocity of the temperature distubance propagation, Cattaneo (1958) has proposed instead of equation (1) a linear hyperbolic equation

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{\rho c}{\lambda}\frac{\partial T}{\partial t}. \tag{2}$$

This equation is known as the telegraph equation, in which $C^2=\lambda/\tau\rho c$ is the propagation velocity of temperature disturbances. When $C=\infty$, equation (2) reduces to equation (1). The analytical solutions to the equation (2) have been presented and discussed by Gembarovic and Mayeriuk (1987, 1988) with respect to the propagation of heat pulses.

The derivation of equation (2) requires a modified Fourier's law including a thermal inertial term $\tau\partial J/\partial T$ associated with the thermal relaxation effect. As a result, the equivalent system of equations corresponding to (2) will be

$$\frac{\partial J}{\partial x}=-\rho c\frac{\partial T}{\partial t} \tag{3}$$

and

$$\tau\frac{\partial J}{\partial t}+J+\lambda\frac{\partial T}{\partial x}=0 \tag{4}$$

where J is the heat flux, ρ the density, c the specific heat, λ the thermal conductivity and τ the relaxation time. When $\tau=0$, equation (4) is Fourier's law

$$J=-\lambda\frac{\partial T}{\partial x}. \tag{5}$$

A great number of publications on the validity of equations (1) and (2) are reported in the literature. It is outside the scope of this paper to review all the publications related to this subject. However, we refer the reader to two excellent review papers by Joseph and Preziosi (1989, 1990) in which the basic aspects related to heat transport by waves are discussed and interpreted. We also refer the reader to a paper by Israel (1987) where the difficulties associated with the traditional theories of heat transfer are analysed.

For the case of harmonic temperature oscillations we have $J=J(x)\,e^{i\omega t}$ and $T=T(x)\,e^{i\omega t}$, so that in the frequency domain equation (4) leads to

$$J=-\lambda^*(\omega)\frac{dT}{dx} \tag{6}$$

which represents the generalized Fourier's law, where $\lambda^*(\omega)$ is expressed by the complex quantity

$$\lambda^*=\lambda(1+i\omega\tau)^{-1}. \tag{7}$$

The relations (6) and (7) reveal that in a regime of temperature oscillations the heat flux is not in phase with the temperature gradient.

As reported in the literature, the violation of equation (5) has been observed in the propagation of thermal waves, where the thermal conductivity is a power law function of temperature expressed as in Zeldovich and Raizer (1967):

$$\lambda(T)=\lambda_0\left(\frac{T}{T_0}\right)^m \tag{8}$$

where λ_0 is the thermal conductivity corresponding to a temperature reference T_0. Note that (8) appears in a real explosion, in which the hot gas is radiating into the ambiant, for example the blast waves in gas dynamics.

Another relevant example of the violation of Fourier's law (5) is the heat transfer in superfluid helium, in which case a nonlinear relation between the heat flux and temperature gradient was found as in the form

$$J=\lambda\left|\frac{\partial T}{\partial x}\right|^{1/3}. \tag{9}$$

This power law relation is known as the Gorter–Melling law. In addition, we mention the case of heat transfer in plasma physics where Fourier's law (5) is no longer valid.

From the considerations shown above, it is evident that there is no justified physical reason why λ in the heat equation (1) should not depend on the temperature, nor why the heat flux J in (5) should not depend on a nonlinear derivative of temperature. Therefore, the heat equation (1) is indeed only an approximation which, however, was found to govern many physical applications.

To evaluate the nonlinear effects associated with violation of equation (1), shown above, we write the system of equations (3) and (4) in the form

$$\frac{\partial J}{\partial x} = -\rho c \frac{\partial T}{\partial t} \tag{10}$$

and

$$\tau \frac{\partial J}{\partial t} + J - \lambda_0 \left(\frac{T}{T_0} \right)^m \left| \frac{\partial T}{\partial x} \right|^n = 0. \tag{11}$$

The case without the relaxation effect, i.e. $\tau = 0$ in (1), leads to nonlinear Fourier's law expressed in the form

$$J = \lambda_0 \left(\frac{T}{T_0} \right)^m \left| \frac{\partial T}{\partial x} \right|^n. \tag{12}$$

It is anticipated that the nonlinear heat model, given by the constitutive equation (12) where three fitting parameters, i.e. λ_0, m and n, are involved, could provide a better agreement between the observed and predicted data for a larger class of heat transfer problems.

From equations (10) and (11) we obtain, by cross-differentiation, the following nonlinear equation of hyperbolic type for determining the transient temperature distribution:

$$\frac{1}{C^2} \frac{\partial^2 T}{\partial t^2} + b \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left[T^m \left| \frac{\partial T}{\partial x} \right|^n \right] \tag{13}$$

where

$$C^2 = \frac{\lambda_0}{\tau \rho c T_0^m} \qquad \text{and} \qquad b = \frac{\rho c T_0^m}{\lambda_0}. \tag{14}$$

For the case $m = 0$ and $n = 1$ equation (2) is recovered, while for $m \neq 0$, $n = 1$ and $C = \infty$ we have from (13)

$$\frac{\partial}{\partial x} \left[T^m \frac{\partial T}{\partial x} \right] = b \frac{\partial T}{\partial t}. \tag{15}$$

This nonlinear diffusion equation was obtained and studied in the paper by Zeldovich and Kompaneets (1959). An exact self-similar solution of equation (15) for the Cauchy problem, given by Zeldovich and Kompaneets (1959), reveals the existence of travelling wave characteristics. We note that this solution is an instantaneous point source solution from which the released heat diffuses in a medium of infinite extent. An equation similar to (15) for the pressure distribution also appears in the polytropic gas flow through a porous medium.

In this paper we are concerned with finding an exact similarity solution to the nonlinear equation

$$\frac{\partial}{\partial x} \left[T^m \left| \frac{\partial T}{\partial x} \right|^n \right] = a^2 \frac{\partial T}{\partial t} \qquad a^2 = \frac{\rho c T_0^m}{\lambda_0} \tag{16}$$

which is derived from (13) without the thermal relaxation effect, i.e. $C = \infty$ in (13).

2. Similarity solutions

To illustrate the nonlinear effects associated with heat equation (16), we investigate here the case of semi-infinite medium, where $\partial T/\partial x<0$, $m\neq 0$ and $n\neq 0$.

The similarity solutions of equation (16) may be obtained by means of the relations

$$\eta = xt^{-\beta} \qquad \text{and} \qquad T=t^{-\alpha}f(\eta). \tag{17}$$

Substituting these relations into (16) we find that $f(\eta)$ satisfies the nonlinear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[f^m\left(-\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^n\right]=a^2\left(\alpha f+\beta\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) \tag{18}$$

provided that between α and β we have the following relation:

$$\beta=\frac{1}{1+n}[1+(1-m-n)\alpha]. \tag{19}$$

The case $\alpha=\beta$ in (18) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[f^m\left(-\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^n\right]=a^2\beta\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta f) \tag{20}$$

and from (19) one has

$$\alpha=\beta=\frac{1}{2n+m}. \tag{21}$$

The first integral of (20) will be

$$f^m\left(-\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^n=a^2\beta\eta f+C \tag{22}$$

C being an integration constant.

The case $\alpha=0$ in (18) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[f^m\left(-\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^n\right]=a^2\beta\eta\frac{\mathrm{d}f}{\mathrm{d}\eta} \tag{23}$$

where

$$\beta=\frac{1}{1+n}. \tag{24}$$

Unfortunately this case is no more integrable in closed form than equation (20), but it is important to emphasize that the nonlinear diffusion equation (16) has been reduced to an ordinary differential equation in a similarity variable. Equation (23) can be solved numerically.

Considering $C=0$ in (22), then $f(\eta)$ is determined from (22) by relation

$$f(\eta)=\left[C-\frac{(m+n-1)(a^2\beta)^{1/n}}{1+n}\eta^{(1+n)/n}\right]^{n/(m+n-1)} \tag{25}$$

where C is also an integration constant. This relation reveals that a moving temperature front exists provided that $m+n>1$, in which case (25) may be rewritten in terms of a new constant η_1 in the form

$$f(\eta)=B[\eta_1^{(1+n)/n}-\eta^{(1+n)/n}]^{n/(m+n-1)} \tag{26}$$

for $\eta < \eta_1$ and $f(\eta) = 0$ for $\eta \geq \eta_1$, where

$$B = \left[\frac{(m+n-1)(a^2\beta)^{1/n}}{1+n}\right]^{n/(m+n-1)}. \tag{27}$$

From previous relations, it can be seen that for $m+n<1$ a temperature front does not exist.

The relations (17) and (26) determine the temperature distribution behind the moving front, which is expressed as

$$T = Bt^{-1/(2n+m)}[\eta_1^{(1+n)/n} - \eta^{(1+n)/n}]^{n/(m+n-1)} \qquad 0<\eta<\eta_1. \tag{28}$$

At $\eta = 0$ we have

$$T(0) = B\eta_1^{(1+n)/(m+n-1)} t^{-1/(2n+m)} \tag{29}$$

so that (28) may be rewritten in dimensionless form

$$\frac{T(\eta)}{T(0)} = \left[1 - \left(\frac{\eta}{\eta_1}\right)^{(1+n)/n}\right]^{n/(m+n-1)} \qquad 0<\eta<\eta_1 \tag{30}$$

and $T(\eta) = 0$ for $\eta \geq \eta_1$.

The temperature distribution corresponding to the case $m+n=1$ may be easily obtained from (25) by means of the relation

$$\lim_{m+n\to 1} C\left[1 - \frac{(m+n-1)(a^2\beta)^{1/n}}{1+n}\eta^{(1+n)/n}\right]^{n/(m+n-1)} = C\exp\left[-\frac{(a^2\beta)}{1+n}\eta^{(1+n)/n}\right]. \tag{31}$$

Since $m+n=1$, then $\eta = xt^{-1/(1+n)}$, so that from (17) and (31) we have

$$T(x,t) = \frac{C}{t^{1/(1+n)}}\exp\left[-(a^2\beta)^{1/n}\frac{x^{(1+n)/n}}{(1+n)t^{1/n}}\right]. \tag{32}$$

For $n=1$ the fundamental solution of the linear diffusion equation is recovered:

$$T(x,t) = \frac{C}{\sqrt{t}}\mathrm{e}^{-a^2x^2/4t}. \tag{33}$$

When $n=1$ and $m>1$, Zeldovich's solution is recovered from (28) and expressed as

$$T = \left[\frac{ma^2}{2(2+m)}\right]^{1/m} t^{-1/(2+m)}(\eta_1^2 - \eta^2)^{1/m} \tag{34}$$

for $0<\eta<\eta_1$ and $T=0$ for $\eta \geq \eta_1$, while from (28) the case $m=0$ and $n>1$ leads to

$$T = \left[\frac{n-1}{n+1}\left(\frac{a^2}{2n}\right)^{1/n}\right]^{n/(n-1)} t^{-1/2n}(\eta_1^{(1+n)/n} - \eta^{(1+n)/n})^{n/(n-1)} \tag{35}$$

for $0<\eta<\eta_1$ and $T=0$ for $\eta \geq \eta_1$. In the above equations η_1 is a constant which will be determined in the next section.

3. The Cauchy problem

We consider the problem when the heat is released instantaneously from a point source located at $x=0$, known as the Cauchy problem. Obviously, from a physical point of view, this problem is appropriate to analyse the nonlinear effects associated with the propagation of temperature waves. For example, the temperature distribution (28) has a blow-up behaviour.

It is well known that the Cauchy problem, i.e. the solution of an instantaneous point source, requires $|\partial T/\partial x|_{x=0}=0$; $l(t)$ is the location of the moving temperature front. From (28) it can be seen that this condition is satisfied. Thus a solution of equation (16) is sought to satisfy the initial condition

$$T(x,0)=Q\delta(x) \qquad \text{or} \qquad \rho c\int_0^\infty T(x,0)\,\mathrm{d}x=Q=\text{constant} \tag{36}$$

and those at the front location $x=l(t)$.

$$T(x\geqslant l(t),t)=0 \qquad \text{and} \qquad \left.\frac{\partial T}{\partial x}\right|_{x\geqslant l(t)}=0 \tag{37}$$

where $\delta(x)$ is the Dirac function and $Q=$ constant the total amount of heat released instantaneously at $x=0$. It should be pointed out that according to (30) the front condition $\partial T/\partial x|_{x=l(t)}=0$ does not hold if $m\geqslant 1$. Therefore, the condition $\partial T/\partial x|_{x=l(t)}=0$ requires $m<1$ and $m+n>1$. The case $m>1$ and $n\neq 1$ leads to $\partial T/\partial x|_{x=l(t)}=\infty$. Despite this fact, however, relations (12), (28) and (30) give a zero heat flux at the front location, i.e. $J(l(t),t)=0$. Figure 1 shows the temperature profiles corresponding to the case $m<1$ and $n=1$.

By integrating (3) over the spatial range $0<x<l(t)$ we obtain

$$J(l(t),t)-J(0,t)=-\rho c\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{l(t)}T\,\mathrm{d}x-T(l(t),t)\frac{\mathrm{d}l}{\mathrm{d}t}\right] \tag{38}$$

which taking into account the conditions (37) becomes

$$J(0,t)=\rho c\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{l(t)}T\,\mathrm{d}x. \tag{39}$$

Since the amount of heat Q may be expressed in terms of heat flux by relation

$$\mathrm{d}Q=J(0,t)\,\mathrm{d}t \tag{40}$$

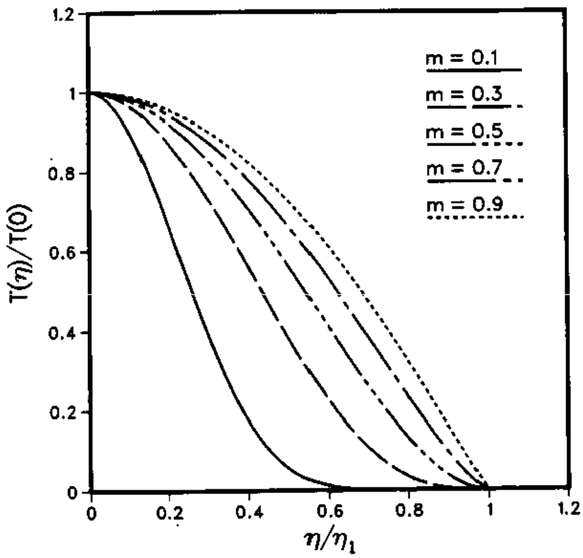

Figure 1. Effect of fitting parameter $m<1$ on the dimensionless temperature distribution for $n=1$.

then we have instead of (36)

$$Q = \rho c \int_0^{l(t)} T \, dx. \tag{41}$$

In terms of similarity variable η the relation (41) is expressed as

$$Q = \rho c \int_0^{\eta_1} f(\eta) \, d\eta$$
$$= \rho c B \eta_1^{1+n/n(m+n-1)} \int_0^{\eta_1} \left[1 - \left(\frac{\eta}{\eta_1} \right)^{(1+n)/n} \right]^{n/(m+n-1)} d\eta. \tag{42}$$

Since $Q =$ constant then from (42) we have

$$Q = \rho c B \frac{n}{1+n} \eta_1^{n(m+n)+1/n(m+n)-1} \Gamma\left(\frac{n}{1+n}\right) \Gamma\left(\frac{m+2n-1}{m+n-1}\right) \left[\Gamma\left(\frac{n}{1+n} + \frac{m+2n-1}{m+n-1} \right) \right]^{-1} \tag{43}$$

where Γ is the gamma function.

The relation (43) determines $\eta_l =$ constant. As a result, the front location is obtained from (17) and (24)

$$l(t) = \eta_1 t^{1/(2n+m)} \tag{44}$$

whereas the propagation velocity of temperature disturbances will be

$$V_f = \frac{dl}{dt} = \frac{\eta_1}{2n+m} t^{(1-2n-m)/(2n+m)}. \tag{45}$$

Since $m + n > 1$, then the front movement is decelerated. For illustrative purposes, we show in figures 2 and 3 the solution behaviour of Cauchy's problem for the dimensionless temperature distribution, given by relation (30), expressed in terms of η/η_1. Figure 2 reveals the effects of the fitting parameter m for $n = 1$, while figure 3 reveals the

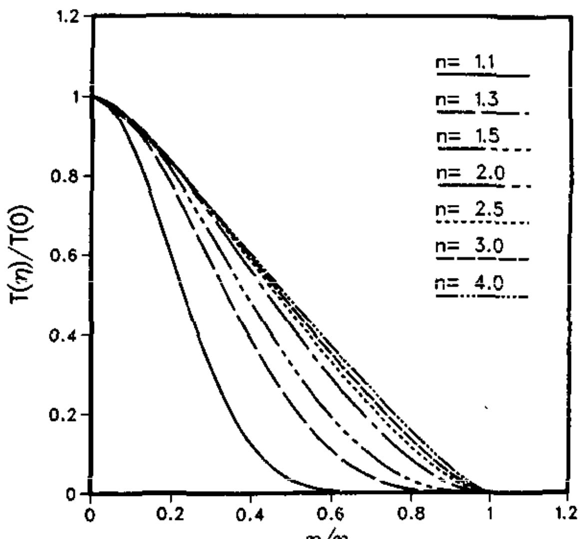

Figure 2. Effects of fitting parameter $n > 1$ on the dimensionless temperature distribution for $m = 1$.

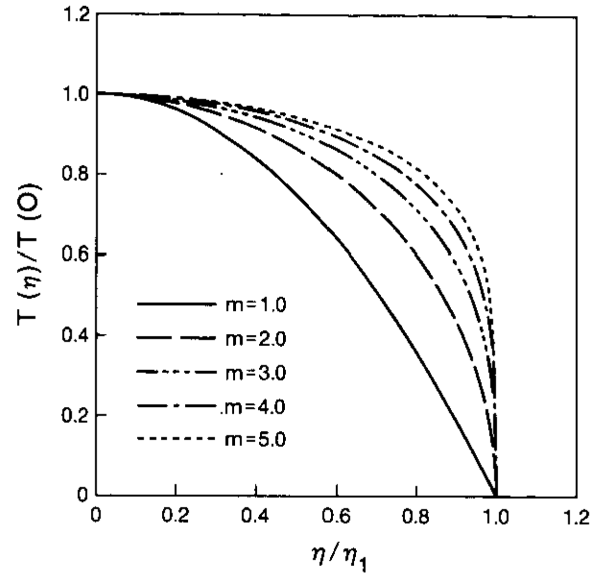

Figure 3. Effects of fitting parameter $m \geqslant 1$ on the dimensionless temperature distribution for $n = 0$.

effects of n for $m \geqslant 1$. From these figures it is evident that m and n have a significant effect on the travelling wave solution of the nonlinear diffusion equation (16), which is a degenerate parabolic equation.

4. Concluding remarks

In this paper we have shown an exact similarity solution in closed form to the nonlinear heat conduction (16) governing the propagation of heat waves.

The solution behaviour, determining the transient temperature response, indicates the existence of travelling wave characteristics. As a result, the temperature disturbances will propagate with finite velocity, which from (45) is a monotonically decreasing function of time. This relevant result is in contrast to the standard linear equation (1), where the temperature disturbances travel with infinite velocity. Consequently, instead of an ill-defined domain of temperature variation, obtained from the linear equation (1), we have from (16) a perfectly defined domain which expands in time according to the relation (44). The considerations shown above point out the main differences between the behaviours of the solutions determined by equations (1) and (16). On the other hand, the parameters m and n determine the shape of the temperature profile, as can be seen from figures 1–3.

Appendix

It is of special interest to illustrate the effects associated with the presence of a source term in equation (16). For this purpose, equation (3) must be written as

$$\frac{\partial J}{\partial x} = -\rho c\left(\frac{\partial T}{\partial t} + Q\right). \tag{A.1}$$

Assuming the linear relation $Q = bT$, where for $b > 0$ we have a sink effect while for $b < 0$ a source effect. As a result, equations (12) and (A.1) lead to the nonlinear diffusion equation

$$\frac{\partial}{\partial x}\left[T^m \left|\frac{\partial T}{\partial x}\right|^n\right] = -a^2\left(\frac{\partial T}{\partial t} + bT\right). \tag{A.2}$$

By means of the function

$$T = \mathrm{e}^{-bt} f \tag{A.3}$$

the equation (A.2) is reduced to

$$\frac{\partial}{\partial x}\left[f^m \left|\frac{\partial f}{\partial x}\right|^n\right] = -a^2 \frac{\partial f}{\partial \tau} \tag{A.4}$$

where

$$\tau = \frac{1}{(m+n-1)b}\left(1 - \mathrm{e}^{-(m+n-1)bt}\right). \tag{A.5}$$

A similarity solution to (A.4) may be obtained by using the relations

$$f = \tau^{-\alpha}\varphi(\eta) \qquad \text{and} \qquad \eta = x\tau^{-\beta}. \tag{A.6}$$

Taking into account that $\partial T/\partial x < 0$, then these relations yield the following differential equation for $\varphi(\eta)$:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\varphi^m\left(-\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}\right)^n\right] = a^2\left(\alpha\varphi + \beta\eta\frac{\mathrm{d}\varphi}{\mathrm{d}\eta}\right) \tag{A.7}$$

provided that

$$\beta(1+n) = 1 + \alpha(1-m-n). \tag{A.8}$$

It is straightforward to show that for the case $\alpha = \beta$ in (A.8) an analytical solution to (A.7) is expressed as

$$\varphi(\eta) = \left[C - \frac{(m+n-1)}{1+n}(a^2\beta)^{1/n}\eta^{(1+n)/n}\right]^{n/(m+n-1)} \tag{A.9}$$

where from (A.8) we obtain

$$\alpha = \beta = \frac{1}{2n+m}. \tag{A.10}$$

Consequently, the relation (A.9) is a travelling wave solution for $m+n>1$. In this case, the previous relations (A.3), (A.6) and (A.9) yield the temperature distribution, in the presence of a sink effect, as follows:

$$T = B\,\mathrm{e}^{-bt}\tau^{-\alpha}\left(\eta_1^{(1+n)/n} - \eta^{(1+n)/n}\right)^{n/(m+n-1)} \tag{A.11}$$

for $\eta < \eta_1$ and $T = 0$ for $\eta \geqslant \eta_1$, where α is given by relation (A.10), τ by relation (A.5) and β by relation (A.10). In the absence of a sink effect, i.e. $b = 0$ in (A.11), we recover the temperature distribution (28).

To determine η_1 in (A.11) for the Cauchy problem, we have (41) which, taking into account (A.11), becomes

$$Q = \rho c B \eta_1^{(1+n)/n(m+n-1)} \, \mathrm{e}^{-bt} \int_0^{\eta_1} \left[1 - \left(\frac{\eta}{\eta_1}\right)^{(1+n)/n}\right]^{n/(m+n-1)} \mathrm{d}\eta. \qquad \text{(A.12)}$$

This relation shows that Q is no longer a constant as in the case $b=0$. This means that a similarity solution to the nonlinear equation (A.2) for the Cauchy problem will require a heat source in which Q has to decline in time according to the relation $Q = Q_0 \, \mathrm{e}^{-bt}$. In this case, the relation (A.12) gives $\eta_1 = \text{constant}$. Based on this fact, the location of the temperature front is determined from (A.5) and (A.6) and expressed as

$$l(t) = \eta_1 \tau^{1/(2n+m)} = \frac{\eta_1}{[(m+n-1)b]^{1/(2n+m)}} [1 - \mathrm{e}^{-(m+n-1)bt}]^{1/(2n+m)} \qquad \text{(A.13)}$$

while the propagation rate of temperature disturbances will be

$$V = \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\eta_1[(m+n-1)b]^{(m+2n-1)/(2n+m)}}{2n+m} \mathrm{e}^{-(m+n-1)bt} [1 - \mathrm{e}^{-(m+n-1)bt}]^{(1-m-2n)/(2n+m)}. \qquad \text{(A.14)}$$

In view of relation (A.13), it turns out that there exists a front location l^* from which the temperature front cannot expand and, consequently, the temperature disturbances cannot be felt for $l > l^*$. For example, considering $\exp(-(m+n-1)bt) \cong 0.01$ in (A.13) then from (A.13) one has

$$l^* = \frac{\eta_1}{[(m+n-1)b]^{1/2n+m}} \qquad \text{(A.15)}$$

which is a direct consequence of the sink effect, i.e. $b > 0$. This relevant result is in contrast to the case $b=0$ in (A.13), where $l(t)$ approaches infinity when $t \to \infty$.

References

Cattaneo C 1958 Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantée. *Compte Rendu Acad. Sci.* **247** 431

Dresner L 1982 *Advances in Cryogenic Engineering* vol 27, ed R W Fost (New York: Plenum) p 411

Gembarovic J and Mayernik V 1987 Determination of thermal parameters of relaxation materials. *Int. J. Heat Mass Transfer* **30** 199

—— 1988 Non-Fourier propagation of heat pulses in finite medium *Inst. J. Heat Mass Transfer.* **31** 1073–80

Joseph D D and Preziosi L 1989 Heat waves *Rev. Mod. Phys.* **61** 41

—— 1990 Addendum to the paper 'heat waves' *Rev. Mod. Phys.* **62** 375

Israel W 1987 *Covariant Fluid Mechanics and Thermodynamics: An introduction* (*Lecture Notes in Mathematics 1385*) (Berlin: Springer)

Pascal H 1989 Propagation of disturbances in a non-Newtonian fluid. *Physica* **39D** 262

Zeldovich Ya and Kompaneets A 1959 *On the Propagation of Heat for Nonlinear Heat Conduction* (in Russian) (Moscow: Izddat Akad. Nauk SSSR) p 61

Zeldovich Ya and Raizer Yu 1967 *Physics of Shock Waves and High-temperature Hydrodyanmic Phenomena* vol II (New York: Academic)